Group A

1. [30 marks: 5+10 +15].

Let $X_1, ..., X_n$ be independent and identically distributed random variables with a uniform distribution on $(0, \theta]$ where $\theta > 0$.

(a) [**5 marks**] Write down the joint probability density function of $X_1, ..., X_n$.

(b) [**10 marks**] Suppose x_i is a realization of X_i, for each $i = 1, ..., n$. And suppose the value of θ is unknown. Find the value of θ that maximizes the joint p.d.f. in part (a) *given* that $x_1, ..., x_n$ have been observed. (This is called the maximum likelihood estimate of θ.)

(c) [**15 marks**] Consider the function: $f(x, y) = x^2 + y^2 - 2x$

(i) Find the maximum value of f over the region $\{(x, y) \mid 2x^2 + 3y^2 - 2x \leq 100\}$

(ii) Find the minimum value of f over the region $\{(x, y) \mid 2x^2 + 3y^2 - 2x \geq 100\}$

2. [30 marks: 3+5+10+6+6]

A *tournament* consists of n players and all possible $C(n, 2) = \frac{n(n-1)}{2}$ pairwise matches between them. There are no ties in a match: in any match, one of the two players wins. The score of a player is the number of matches she wins out of all her $(n-1)$ matches in the tournament. Denote the score vector of the tournament as $s \equiv (s_1, \dots, s_n)$ and assume without loss of generality $s_1 \geq s_2 \geq \dots \geq s_n$.

(a) **(3 marks)** For any $2 \leq k \leq n$, show that $s_1 + \dots + s_k \geq C(k, 2)$, where $C(k, 2) = \frac{k(k-1)}{2}$.

(b) **(5 marks)** Suppose $n > 3$ and players $1, 2, 3$ win every match against players in $\{4, \dots, n\}$. Find the value of $s_4 + \dots + s_n$?

(c) **(10 marks)** Suppose $s_n = s_0, s_{n-1} = s_0 + 1, s_{n-2} = s_0 + 2$ for some positive integer s_0 and $n \geq 3$. Show that

$$s_0 \leq \frac{(n-2)(n-3)}{2n}.$$

(d) **(6 marks)** A tournament generates a score vector s such that

$$s_j - s_{j+1} = 1 \text{ for all } \ j \in \{1, \dots, n-1\}.$$

What is the score vector of this tournament? For every Player j, who does Player j beat in this tournament?

(e) **(6 marks)** Suppose there are six players, i.e., $n = 6$. There is a tournament such that each player has a score of at least two and difference in scores of any two players is not more than one. What is the score vector of this tournament? Construct a tournament (describing who beats who) which generates this score vector.

3. [30 marks: 6+6+6+4+8]

Consider the following equation in x:

$$(x-1)(x-2)\cdots(x-n) = k, \tag{1}$$

where $n > 1$ is a positive integer and k is a real number. Argue whether the following statements are true or false by providing a proof or a counter example.

(a) **(6 marks)** Suppose $n = 2$. There is a real solution to Equation (1) for every value of k.

(b) **(6 marks)** Suppose $n = 3$. There is a real solution to Equation (1) for every value of k.

(c) **(6 marks)** For all $k \geq 0$ and for every positive integer $n > 1$, there is a real solution to Equation (1).

(d) **(4 marks)** For all $k < 0$ and for every *odd* positive integer $n > 1$, there is a real solution to Equation (1)

(e) **(8 marks)** For all $k < 0$, there is *some even* positive integer n such that a real solution to Equation (1) exists.

Group B

1. Consider an economy inhabited by identical agents of size 1. A representative agent's preference over consumption (c) and labour supply (l) is given by the utility function

$$u(c,l) = c^{\alpha}(24-l)^{1-\alpha},\ 0<\alpha<1.$$

Production of the consumption good c is given by the production function $c = Al$, where $A > 0$ is the productivity of labour. Both the commodity market and labour market are perfectly competitive: the buyers and sellers take the price as given while taking demand and supply decisions. Let us denote the hourly wage rate by $w > 0$ and price of the consumption good by $p > 0$.

(a) [**25 marks: 13 + 7 + 5**] Competitive Equilibrium:

A competitive equilibrium is given by the allocation of consumption and labour, (c^{CE}, l^{CE}), and the relative price ratio, $\frac{w}{p}$, such that, given w and p, a representative agent decides her labour supply, l^S, and consumption demand, c^D, to maximize her utility; a firm decides its labour demand, l^D, and supply of consumption good, c^S, to maximize its profit; and, finally, both the commodity market and labour market clear, that is, $l^D = l^S$ and $c^D = c^S$.

(i) [**13 marks**] Set up the representative agent's utility maximization problem. Write down the first order conditions for this maximization problem and determine l^S and c^D as functions of w and p.

(ii) [**7 marks**] Set up a firm's profit maximization problem. Determine l^D and c^S as functions of w and p.

(iii) [**5 marks**] Determine the competitive equilibrium allocation, (c^{CE}, l^{CE}), and the relative price ratio, $\frac{w}{p}$.

(b) [**5 marks**] Pareto efficient allocation:

For this economy define the concept of a Pareto efficient allocation of consumption and labour. Find out a Pareto efficient allocation of consumption and labour in this economy. Provide a clear explanation.

2. [30 marks = 12 + 18]

(a) [**12 marks**] Ms. A's income consists of Rs.1,00,000 per year from pension plus the earnings from whatever she sells of the 2,000 kilograms of rice she harvests annually from her farm. She spends this income on rice (x) and on all other expenses (y). All other expenses (y) are measured in rupees, so that the price of y is Rs. 1. Last year rice was sold for Rs. 20 per kilogram, and Ms. A's rice consumption was 2,000 kilograms, just the amount produced on her farm. This year the price of rice is Rs. 30 per kilogram. Ms. A has standard convex preferences over rice and all other expenses. *Answer the following two questions without referring to any utility function or indifference curves.*

(i) [**7 marks**] What will happen to her rice consumption this year – increase, decrease, or remain the same? Give a clear explanation for your answer.

(ii) [**5 marks**] Will she be better or worse off this year compared to last year? Explain clearly.

(b) [**18 marks**] There are two goods x and y. Mr. B has standard convex preferences over the two goods. He has endowments of $e_x > 0$ units of good x and $e_y > 0$ units of good y. He does not have any other source of income. When the price of good y is Rs. 1 and the price of good x is Rs. p_x, he decides neither to buy nor to sell good x.

(i) [**8 marks**] Suppose that, for good x, the prices have become Rs. $p_L < p_x$ if an individual is a *seller* and Rs. $p_H > p_x$ if an individual is a *buyer*. The price of good y remains Rs. 1 no matter whether an individual buys or sells good y. Write down the equation of the new budget constraint and draw it labelling the important points clearly.

(ii) [**10 marks**] Will Mr. B buy or sell good x? By how much? Give a clear explanation for your answer without referring to any utility function or indifference curves.

3. [30 marks = 6+7+7+10]

Consider a moneylender who faces two types of potential borrowers: the *safe type* and the *risky type*. Each type of borrower needs a loan of the same size L to invest in some project. The borrower can repay only if the investment produces sufficient returns to cover the repayment. Suppose that the safe type is always able to obtain a secure return of R from the investment, where $R > L$. On the other hand, the risky type is an uncertain prospect; he can obtain a higher return R' (where $R' > R$), but only with probability p. With probability $1-p$, his investment backfires and he gets a return of 0. The money lender has enough funds to lend to just one applicant, and there are two of them – one risky, one safe. Each borrower knows his own type, but the moneylender does not know the borrower's type. He just knows that one borrower is a safe type and the other one is a risky type. Since the moneylender has enough funds to lend to just one applicant, when both the borrowers apply for the loan, he gives the loan randomly to one of them, say by tossing a coin. Assume that the lender supplies the loan from his own resources and his opportunity cost is zero.

(a) [**6 marks**] What is the highest interest rate, call it i_s, for which the safe borrower wants the loan? What is the highest interest rate, i_r, for which the risky borrower wants the loan? Who is willing to pay a higher interest rate, the risky borrower or the safe borrower?

(b) [**7 marks**] The lender's objective is to maximize his expected profit. Argue clearly that the lender's effective choice is between two interest rates, i_s and i_r. (That is, argue that the lender will *not* choose any interest rate strictly lower than $\min\{i_r, i_s\}$, any interest rate strictly higher than $\max\{i_r, i_s\}$, or any interest rate strictly in between i_r and i_s.)

(c) [**7 marks**] Argue that when the lender charges i_r, his expected profit is given by $p(1+i_r)L - L$. Derive, with a clear argument, the expression of lender's expected profit when he charges i_s.

(d) [**10 marks**] An equilibrium with *credit rationing* occurs when, at the equilibrium interest rate, some borrowers who want to obtain loans are unable to do so; however, lenders do not raise the interest rate to eliminate the excess demand.

Explain clearly that we have an equilibrium with *credit rationing* when

$$p < \frac{R}{2R' - R}.$$

Group C

1. [30 marks=3+8+2+10+4+3]

Consider an economy where identical agents (of mass 1) live for two periods: youth (period 1) and old age (period 2). The utility function of a representative agent born at time t is given by

$$u(c_{1,t}, c_{2,t+1}) = \log(c_{1,t}) + \beta \log(c_{2,t+1}),$$

where c_1 denotes consumption in youth, c_2 denotes consumption in old age, and $0 < \beta < 1$ is the discount factor reflecting her time preference. In her youth the representative agent supplies her endowment of 1 unit of labour inelastically and receives the market-determined wage rate w_t. So in her youth the agent faces the budget constraint $c_{1,t} + s_t = w_t$, where s_t denotes her savings. When old, she just consumes her savings from youth plus the interest earning on her savings, $s_t r_{t+1}$, where r_{t+1} is the market-determined interest rate in period $t+1$. That is, when old, her budget constraint is $c_{2,t+1} = (1 + r_{t+1}) s_t$.

(a) [**3 marks**] Set up the agent's utility maximization problem by showing her choice variables clearly.

(b) [**8 marks**] Write down the first order conditions for this maximization problem and derive the savings function. Explain how savings, s_t, if it does, depends on the interest rate r_{t+1}.

The production function of the economy is given by $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, $0 < \alpha < 1$, where K and L denote the amounts of capital and labour in the economy, respectively. Capital depreciates fully after use, that is, the rate of depreciation of capital is one. Factor markets being competitive, the equilibrium factor prices are given by their respective marginal products.

(c) [**2 marks**] Derive the equilibrium wage rate (w_t) of the economy in terms of K_t. [Keep in mind that the mass of agents is 1 and each agent supplies her endowment of 1 unit of labour inelastically.]

The role of the financial sector (banks, stock market, and so on) is to mobilize the savings of households to bring it for effective use by the production sector. But the financial sector does not work well and a fraction $0 < \theta < 1$ of aggregate savings gets lost (vanishes in thin air) in the process of intermediation.

(d) [**10 marks**] Derive the law of motion of capital (that is, express capital in period $t+1$, K_{t+1}, in terms of capital in period t, K_t).

(e) [**4 marks**] Derive the steady state amount of capital of the economy.

(f) [**3 marks**] How does the steady state amount of capital depend on the inefficiency of the financial sector θ?

2. [30 marks = 5+5+10+10]

Consider a Solow-Swan model with learning by doing. Assume that the production function is of the form

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

where A is the level of technological progress and grows at the rate $g > 0$, L is the population with grows at the rate $n > 0$, K is the capital stock, Y is GDP, and $\alpha \in [0, 1]$. Assume that

$$\dot{K} = sY - \delta K$$

Define $Z = \frac{K}{AL}$ as the capital labor ratio in efficiency units. Let output per worker be given by $Q = AZ^{\alpha}$. The parameter $s \in [0, 1]$ denotes the savings rate. The parameter $\delta \in [0, 1]$ denotes the depreciation on capital.

(a) [**5 marks**] Derive an expression for $\frac{\dot{Z}}{Z}$

(b) [**5 marks**] Instead of assuming that the rate of technological progress is constant (g), now assume that the instantaneous increase in A is proportional to output per worker, i.e., there is learning by doing

$$\dot{A} = \gamma Q.$$

Show that the law of motion of capital is given by

$$\dot{Z} = (s - \gamma Z)Z^{\alpha} - (\delta + n)Z$$

(c) [**10 marks**] Draw a diagram describing the dynamics of growth in the model with learning by doing. Plot Z on the $x - axis$, and the appropriate functions on the $y - axis$

(d) [**10 marks**] In contrast to the model with no learning by doing, does an increase in the investment rate raise the balanced-growth rate ? What does this tell you about the change in policy having level effects versus growth effects in the model with learning by doing in contrast to the model when there is no learning by doing ? Show your answer using the diagram in part (c).

3. [30 marks =10+10+3+3+4]

Suppose households who live till T periods maximize $\sum_{n=t}^{T} \beta^{n-t} \ln(c_n)$ where c_n represents their income in period $n = t, t+1, t+2, ...T$ and β is a parameter with $0 < \beta < 1$. Suppose per period income and the saving of households are y_n and s_n respectively and the activity starts from the beginning of their life t. Further, the net interest rate on saving in between any two periods is exogenously fixed at r and so the gross rate of return is $1 + r$. Households have only two activities in every period - consuming and saving.

(a) [**10 marks**] Write down the sequence of budget constraints (one for each period) and the aggregate budget constraint derived from these periodic budget constraints where on the left hand side, consumption levels for all the periods appear, and on the right hand side, income in all periods appears.

(b) [**10 marks**] Under what condition between β and r, is the optimal solution for the above problem yield constant consumption, $\widetilde{C}$, in every period ?

(c) Suppose the condition that you derive in (b) holds. Then answer the following questions in (c) and (d)

- **(i)** [**3 marks**] For a transitory change in income in period t only, calculate the change in the constant level of consumption, $\widetilde{C}$.
- **(ii)** [**3 marks**] For a transitory change in income in period $t + k$ only, calculate the change in the constant level of consumption, $\widetilde{C}$.

(d) [**4 marks**] For a permanent change in income (assume the same amount of income change in all periods), calculate the change in the constant level of consumption, $\widetilde{C}$. Compare this value derived with part c (i).